Generic Quantum Markov Semigroups With Instantaeous states

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We construct a generic quantum Markov semigroup with instantaneous states exploiting the invariance of the diagonal algebra and the explicit form of the action of the pre-generator on off-diagonal matrix elements. Our semigroup acts on a unital C^*- algebra and is strongly continuous on this algebra (Feller property). We discuss the generic hydrogen type atoms as an example.

This class of quantum Markov semigroups is very interesting, not only because it is very big, but also for its rich structure arising from the investigations by Accardi, Hachicha and Ouerdiane [4]; Accardi, Fagnola and Hachicha [3] ; Carbone, Fagnola and Hachicha [17]; Hachicha [18]; Ghorbel, Fagnola, Hachicha, Ouerdiane [7] .

In this talk we construct the generic quantum Markov semigroup (with a finite number of instantaneous states) describing the evolution of a generic system interacting with a Boson, positive temperature, gauge invariant reservoir (The zero temperature case has been studied by A. Ben Ghorbel, F. Fagnola, S. Hachicha, and H. Ouerdiane [7]).
We show that, under suitable continuity (see **GS1 GS2**) on the transition rates of the underlying classical Markov process, it is possible to construct a quantum Markov semigroup, strongly continuous on a C^*-algebra $\mathcal{A}$, whose generator arising from the stochastic limit on a dense domain.
We start our investigations by describing generic quantum Markov semigroups in Sect. 2 and show that, when the system Hamiltonian H_S is has the same type of the hydrogen atom, then instantaneous states appear (Sect. 3).

In Sect. 4 we construct the associated classical Markov semigroup on the invariant Abelian algebra. The extension to the whole C^*-algebra $\mathcal{A}$ is done in Sect. 5.

1 The generic QMS

Let S be the discrete system with the Hamiltonian

$$H_S = \sum_{\sigma \in V} \varepsilon_\sigma \left| \sigma \right\rangle \left\langle \sigma \right| \tag{1}$$

where V is a finite or countable set, $(\left| \sigma \right\rangle)_{\sigma \in V}$ is an orthonormal basis of the complexe seperable Hilbert space h of the system. We call the Hamiltonian H_S is *generic* [5], i.e. if the eigenspace associated with each eigenvalue ε_σ in one dimensional and one has $\varepsilon_\sigma - \varepsilon_{\sigma'} = \varepsilon_\tau - \varepsilon_{\tau'}$ for $\sigma \neq \sigma'$ if and only if $\sigma = \tau$ and $\sigma' = \tau'$.

Quantum Markov semigroup was obtained by Accardi and Kozyrev [5] (see also the book [6]) in the stochastic limit of a discrete system with generic free Hamiltonian H_S interacting with a mean zero, gauge invariant, Gaussian field. The interaction between the system and the field has the dipole type form

$$H_I = D \otimes A^+ (g) + D^+ \otimes A (g) ,$$

where D, D^+ are the system operators, i.e. an operator on h, with domain $\mathcal{V}$ (i.e.$\mathcal{V}$ the linear manifold generated by finite combinations of vector $|\sigma\rangle$), such that $\langle u, Dv\rangle = \langle D^+u, v\rangle$, for u, $v \in \mathcal{V}$ and satisfying the analyticity condition

$$\sum_{n=1}^{+\infty} \frac{|\langle \sigma', D^n \sigma\rangle|}{\Gamma(\theta n)}$$

where Γ is the gamma Euler function, for all σ, $\sigma' \in V$ and some $\theta \in]0,1[$, and $A^{+}(g)$, $A(g)$ are the creation and annihilation operators on the Boson Fock space over a Hilbert space with test function $g \in L^2\left(R^d\right)$, $d \geq 3$. The function g is called the form factor (or cutoff) of the interaction and will be supposed to belong to the shawrz space of smooth function rapidly decreasing at infinity.

The form generator of the generic quantum Markov semigroup ([17])

$$\begin{aligned} \pounds(x) &= \frac{1}{2} \sum_{\substack{\sigma,\sigma' \in V \\ \varepsilon_\sigma \neq \varepsilon_{\sigma'}}} \big(\gamma_{\sigma\sigma'} (2 \left|\sigma\right\rangle \left\langle\sigma' \left|x\right| \sigma'\right\rangle \left\langle\sigma\right| - \{ \left|\sigma\right\rangle \left\langle\sigma\right|, x\}) \\ &\quad + i\xi_{\sigma\sigma'} \left[x, \left|\sigma\right\rangle \left\langle\sigma\right|\right]\big) \end{aligned} \tag{2}$$

Sometimes it will be convenient to distinguish these coefficients in terms of the order of the eigenvalues (energies) associated with the indexes σ, σ' and we will denoted, for $\varepsilon_\sigma > \varepsilon_{\sigma'}$,

$$\gamma^{-}_{\sigma\sigma'} = \gamma_{\sigma\sigma'}, \qquad \xi^{-}_{\sigma\sigma'} = \xi_{\sigma\sigma'}, \qquad \gamma^{+}_{\sigma'\sigma} = \gamma_{\sigma'\sigma}, \qquad \xi^{+}_{\sigma'\sigma} = \xi_{\sigma'\sigma}.$$

In other word we add a $-$ (resp $+$), to $\gamma_{\sigma\sigma'}$ and $\xi_{\sigma\sigma'}$ to stress that it is the rate of a transition to a lower (resp upper) level.

In our model the constants $\gamma^{-}_{\sigma\sigma'}$, $\xi^{-}_{\sigma\sigma'}$, $\gamma^{+}_{\sigma'\sigma}$, $\xi^{+}_{\sigma'\sigma}$ are given by (see [5])

$$\gamma^{-}_{\sigma\sigma'} = 2Re\,(g \mid g)^{-}_{\omega} \left|\langle\sigma', D\sigma\rangle\right|^2, \; \xi^{-}_{\sigma\sigma'} = Im\,(g \mid g)^{-}_{\omega} \left|\langle\sigma', D\sigma\rangle\right|^2, \tag{3}$$

$$\gamma^{+}_{\sigma'\sigma} = 2Re\,(g \mid g)^{+}_{\omega} \left|\langle\sigma', D\sigma\rangle\right|^2, \; \xi^{+}_{\sigma'\sigma} = Im\,(g \mid g)^{+}_{\omega} \left|\langle\sigma', D\sigma\rangle\right|^2, \tag{4}$$

with $\omega = \varepsilon_\sigma - \varepsilon'_\sigma, \qquad \varepsilon_\sigma > \varepsilon_{\sigma'}$.

The constants $(g \mid g)^{-}_{\omega}$, $(g \mid g)^{+}_{\omega}$ are given by

$$(g \mid g)^{-}_{\omega} = \pi \int_{S(\omega)} \frac{|g(k)|^2 e^{\beta\omega(k)}}{e^{\beta\omega(k)} - 1} d_S k - iP.P. \int_{R^d} \frac{|g(k)|^2 e^{\beta\omega(k)}}{(e^{\beta\omega(k)} - 1)(\omega(k) - \omega)} dk$$

$$(g \mid g)^+_\omega = \pi \int_{S(\omega)} \frac{|g(k)|^2}{e^{\beta\omega(k)} - 1} d_S k - iP.P. \int_{R^d} \frac{|g(k)|^2}{(e^{\beta\omega(k)} - 1)(\omega(k) - \omega)} dk$$

where (see [[5], p.15]) $\beta > 0$ is the inverse temperature, $S(\omega)$ denotes the surface in R^d,
$S(\omega) = \{k \in |\omega(k) = \omega\}$, $d_S k$ the surface integral and P.P. the principal part of the integral.
It is well-known from the classical theory of Markov jump processes that the construction of the semigroup from generator with jump intensities $\gamma_{\sigma\sigma'}$ can be done by the standard minimal semigroup method (see [12] [14]) under the following summability condition

$$\mu_\sigma := \sum_{\sigma', \varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma^-_{\sigma\sigma'} = \sum_{\sigma' \varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} \tag{5}$$

and

$$\lambda_\sigma := \sum_{\sigma', \varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma^+_{\sigma\sigma'} = \sum_{\sigma' \varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'} \tag{6}$$

The next section shows a class models, based on hydrogen type atoms, in which the summability conditions does not hold. These models motivate the following assumptions that will be in force throughout this talk:
HS The set $\mathbf{V}$ of eigenvalues of the system Hamiltonian H_S is compact and has a discrete set of accumulation points $(a_j)_{j\geq 1}$ (resp $(b_k)_{k\geq 1}$) belonging to $\mathbf{V}$ that are left (resp right)accumulation points, isolated from the right (resp left) in $\mathbf{V}$ and the series $\mu_\sigma = \sum_{\sigma' \varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'}$ (resp $\lambda_\sigma = \sum_{\sigma' \varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'}$) are absolutely convergent for all $\sigma \in V$ such that $\varepsilon_\sigma \in \mathbf{V} \backslash \{a_1, \cdots, a_n\}$ (resp $\varepsilon_\sigma \in \mathbf{V} \backslash \{b_1, \cdots, b_m\}$).

Therefore, throughout this paper, we shall assume that the following summability condition

$$\sum_{\sigma' \in V, \epsilon_{\sigma'} < \epsilon_\sigma} |\xi^-_{\sigma\sigma'}| + |\xi^+_{\sigma'\sigma}| < \infty \tag{7}$$

holds for all $\sigma \in V$ and define

$$\kappa_\sigma := \sum_{\sigma' \in V, \epsilon_{\sigma'} < \epsilon_\sigma} \xi^-_{\sigma\sigma'} + \xi^+_{\sigma'\sigma}$$

Clearly a_j (resp b_k) is a left (resp right) accumulation point if it is the limit of an increasing (resp decreasing) sequence of eigenvalue of H_S but there is a smallest eigenvalue of H_S strictly bigger (resp upper) than a_j (resp b_k), unless a_j (resp b_k) is the maximum (resp minimum) eigenvalue of H_S.
Denote by $\mathbf{A}$ (resp $\mathbf{B}$) the set of accumulation points a_j (resp b_k) and note that, since it is discrete, it has no accumulation points.
Note that, since all the a_j (resp b_k) belong to $\mathbf{V}$, the spectrum of H_S is compact and bounded from above.

This technical assumption could be removed by slight modifications of the construction in section 4.
Under the above assumption it is clear that the form generator 2 is well defined on the algebra generated by finite rank operators of the form $|\sigma'\rangle\langle\sigma|$ with $\varepsilon_{\sigma'}$, $\varepsilon_{\sigma} \in \mathbf{V}\backslash(\mathbf{A} \cup \mathbf{B})$.

The divergence of the series defining μ_{a_j} (resp λ_{b_k}) means that a_j (resp b_k) is an instantaneous state of the classical Markov process associated with the restriction of $\pounds$ to the Abelian algebra of the operators commuting with the system Hamiltonian H_S and the sojourn time of the process in a_j (resp b_k) is zero.

2 Hydrogen Type Atoms

Hydrogen type atoms provide examples satisfying the previous hypothesis with appropriate conditions on D and the cutoff g.

The hydrogen atom Hamiltonian is the self-adjoint operator H on the Hilbert space $L^2(R^3)$ given by

$$H = -\triangle - \frac{2}{r} \tag{8}$$

where $r = \|x\|$ with $x \in R^3$. In spherical coordinates H is given by

$$H = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r} \tag{9}$$

For each $n \geq 1$ there exist a unique radial solution of the equation

$$H\psi_n = -\frac{1}{n^2}\psi_n \tag{10}$$

With unit norm in $L^2(R^3)$ given by

$$\psi_n(r) = \frac{1}{\sqrt{2}n^{5/2}} e^{-r/2n} L^1_{n-1}(\frac{r}{n}) \tag{11}$$

where L^1_{n-1} is the laguerre polynomial (see [1]).

The operator H is not generic. Indeed, taking two distinct Pythagorean triples as, for instance $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, multiplying them by 13^2 and 5^2 we find $39^2 + 52^2 = 25^2 + 60^2$. Finally, some obvious manipulations yields

$(25.52.60)^{-2} - (39.52.60)^{-2} = (25.39.52)^{-2} - (25.39.60)^{-2}$.

We know from L. Accardi [2] that it is also possible to classify the degree on degeneracy of this spectrum.

A small perturbation of this operator, however, leads to a generic Hamiltonian.

Let $\tau \in]0,1[$ be a transcendental number and, for $n \in V := N^* \cup \{\infty\}$, we define

$$\varepsilon_n = -(n+\tau)^{-2}, \varepsilon_\infty = 0 \tag{12}$$

Now, since τ can not be a solution to any polynomial equation with integer coefficients it is clear that the operator H_S with the above choice of the ε_n is generic.
If g is a square integrable radial function on R^3 such that $g(r) = r^{-\frac{\theta}{2}}$, for $r = \|k\| < 1$, with $\theta < 2$, for any n, $m \in N$ with $n > m$, we have

$$\gamma^+_{nm} = \frac{4\pi^2}{(e^{\beta(\varepsilon_n - \varepsilon_m)} - 1)(1-\theta)} \left(\frac{1}{(m+\tau)^2} - \frac{1}{(n+\tau)^2} \right)^{(1-\theta)} |\langle \psi_m, D\psi_n \rangle|^2,$$

$$\gamma^-_{mn} = e^{\beta(\varepsilon_n - \varepsilon_m)} \gamma^+_{nm}, 1 \leq m < n < \infty$$

and for any $m \in N$

$$\gamma^{+}_{\infty m} = \frac{4\pi^2}{(e^{-\beta\varepsilon_m} - 1)(1-\theta)} \frac{1}{(m+\tau)^{2-2\theta}} |\langle \psi_m, D\psi_\infty \rangle|^2,$$

$$\gamma^{-}_{m\infty} = e^{-\beta\varepsilon_m} \gamma^{+}_{\infty m}$$

For any $n \in N$ there are only a finite number of levels $m < n$, therefore the series $\sum_{m<n} \gamma^{-}_{nm}$ defining μ_n is reduced to a finite sum.
Suitable choices of the operator D yield generalised susceptivities
$\sum_m \gamma^{-}_{m\infty} = \infty = \mu_\infty$. We can take, for instance, a D such that
$|\langle \psi_m, D\psi_n \rangle|^2 = (\frac{1}{(m+\tau)^2} - \frac{1}{(n+\tau)^2})^\alpha$, $m < \infty$
$|\langle \psi_m, D\psi_\infty \rangle|^2 = \frac{1}{(m+\tau)^{2\alpha}}$, $m < \infty$
for some $\alpha > \frac{1}{2}$ to ensure that $\sum_m |\langle \psi_m, D\psi_\infty \rangle|^2 = \|D\psi_\infty\| < \infty$, to find $\mu_\infty = \sum_{n\geq 1} \gamma^{-}_{n\infty} = \infty$ for $3 < 2 + 2\alpha < 2\theta + 1 < 5$.
So we have usually $\lambda_\infty = 0$ and $\lambda_n = \infty \; \forall n \in N$.
This, however, is just an example where our results apply.

3 The QMS on the diagonal algebra

The construction of the Markov semigroup of a classical jump process with some instantaneous states is usually a difficult problem that has not a general solution (see the book [13] ch.7p.376). Even the case of a single instantaneous state is also non trivial (see the papers [12], [16]). Motivated by the discussion of hydrogen type atoms in section 3, however, we give a solution to this problem for a classical which, to the best our knowledge, is new also for classical probability.

The natural definition of the generator L

$$\begin{aligned}
(Lf)(\varepsilon_\sigma) &= \sum_{\sigma', \varepsilon_\sigma \neq \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'}(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma)) \\
&= \sum_{\sigma', \varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'}(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma)) + \sum_{\sigma', \varepsilon_\sigma > \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'}(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma))
\end{aligned}$$

might not may sense for $\varepsilon_\sigma = a_j$ (resp $\varepsilon_\sigma = b_k$) if the function f is not continuous at a_j (resp b_k).
Indeed, if $f(\varepsilon_{\sigma'}) > f(\varepsilon_\sigma) + c$ (c positive constant) in a left (resp a right) neighborhood of a_j (resp b_k), then the series is divergent.

To circumvent this problem we start by restricting the candidate domain of the operator L to function which are left (resp right) continuous at points in **A** (resp in **B**).
Therefore we work in the Banach space $\mathcal{C}$ of complex-valued continuous functions on the set **V** (endowed with the topology induced by the Euclidean topology of R) of eigenvalues with norm $\| \, . \, \|_\infty$ defined by

$$\|f\|_\infty = \sup_{\sigma \in V} |f(\varepsilon_\sigma)|$$

we assume that the generalised susceptivities satisfy the following condition which is necessary for the generator L to be defined at least on indicator functions of isolated points of $\mathbf{V}$ and functions constant in a left (resp right) neighborhood of accumulation points $a_1, \cdots, a_n$ (resp $b_1, \cdots, b_m$) (see Lemma 4-1 below).

GS1

1- For all $\varepsilon_\sigma \in \mathbf{V} \backslash \mathbf{A}$ and $a_j \in \mathbf{A}$ with $a_j > \varepsilon_\sigma$ we have

$$\lim_{\varepsilon_\tau \to a_j^-} \gamma_{\tau\sigma} = \gamma_{a_j\sigma} \tag{13}$$

For all accumulation point a_j, all ε_σ with $a_{j-1} < \varepsilon_\sigma < a_j$ (with the convention $a_{-1} = -\infty$), and all $a_k > a_j$ we have

$$\lim_{\varepsilon_\tau \to a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\tau\sigma'} = \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{a_j\sigma'}; \lim_{\varepsilon_\tau \to a_k^-} \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq a_j} \gamma_{\tau\sigma'} = \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq a_j} \gamma_{a_k\sigma'} \tag{14}$$

2- For all $\varepsilon_\sigma \in \mathbf{V}\backslash\mathbf{B}$ and $b_k \in \mathbf{B}$ with $\varepsilon_\sigma > b_k$, we have

$$\lim_{\varepsilon_\tau \to b_k^+} \gamma_{\sigma\tau} = \gamma_{\sigma b_k}. \tag{15}$$

For all accumulation point b_k, all ε_σ with $b_k < \varepsilon_\sigma < b_{k+1}$, and all $b_l < b_k$ we have

$$\lim_{\varepsilon_\tau \to b_k^+} \sum_{\varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma'\tau} = \sum_{\varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma' b_k}; \lim_{\varepsilon_\tau \to b_l^+} \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq b_l} \gamma_{\sigma'\tau} = \sum_{\varepsilon_\sigma \leq \varepsilon_{\sigma'} \leq b_l} \gamma_{\sigma' b_l} \tag{16}$$

3- we suppose that $\mathbf{A} \cap \mathbf{B} = \emptyset$.

Remark 3.1 *These assumptions clearly holds for the generalised susceptivities of the hydrogen type atoms (section 3).*

We shall prove that the operator $L : D(L) \rightarrow \mathcal{C}$

$$\begin{aligned} D(L) &= \{f \in \mathcal{L}^{\infty}(V) | \exists \lim_{\varepsilon_\sigma \to a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} \left(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma) \right) \forall j \\ & \quad \textit{and} \quad \exists \lim_{\varepsilon_\sigma \to b_k^+} \sum_{\varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'} \left(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma) \right) \forall k \}, \end{aligned}$$

$$(Lf)(\varepsilon_\sigma) = \sum_{\varepsilon_\sigma \neq \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'} \left(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma) \right), \forall \varepsilon_\sigma \in \mathbf{V} \backslash (\mathbf{A} \cup \mathbf{B})$$

$$\begin{aligned} (Lf)(a_j) &= \lim_{\varepsilon_\sigma \to a_j^-} \sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} \left(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma) \right) \\ & \quad + \sum_{\varepsilon_{\sigma'} > a_j} \gamma_{a_j\sigma'} \left(f(\varepsilon_{\sigma'}) - f(a_j) \right) \end{aligned}$$

$$
\begin{aligned}
(Lf)(b_k) &= \lim_{\varepsilon_\sigma \to b_k^+} \sum_{\varepsilon_\sigma < \varepsilon_{\sigma'}} \gamma_{\sigma\sigma'} \left(f(\varepsilon_{\sigma'}) - f(\varepsilon_\sigma) \right) \\
&+ \sum_{\varepsilon_{\sigma'} < b_k} \gamma_{b_k\sigma'} \left(f(\varepsilon_{\sigma'}) - f(b_k) \right)
\end{aligned}
$$

generates a strongly continuous contraction semigroup on $\mathcal{C}$.
As a first step we prove the following

Lemma 3.2 *Suppose that the hypothesis **GS1** holds. Then L is densely defined and closed.*

Lemma 3.3 *The operator L is dissipative.*

Lemma 3.4 *For all $f \in D(L)$ and $\alpha > 0$ we have*

$$
\|\alpha f - Lf\| \geq \alpha \|f\|
$$

We introduce now our last hypothesis on the generalised susceptivities in order to check that the range of $\alpha - L$ is the whole $\mathcal{C}$.

GS2...

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Proposition 3.5 *For all $\alpha > 0$ the range of the operator $\alpha - L$ is the whole $\mathcal{C}$.*

Proof. Open problem

Summing up we proved the following

Theorem 3.6 *The operator L generates a strongly continuous contraction semigroup $(\mathrm{T}_t)_{t\geq 0}$ on $\mathcal{C}$.*

Theorem 3.7 *The semigroup $(\mathrm{T}_t)_{t\geq 0}$ is Markov.*

4 Extension to non diagonal operators

In this section we construct the generic Quantum Markov Semigroup with instantaneous states on a $\mathcal{C}^*-$subalgebra $\mathcal{A}$ of $\mathcal{B}(h)$. The structure 2 of the form generator makes it naturel the following definition of the action of the QMS on off-diagonal operator $|\sigma\rangle\langle\tau|$ with $\varepsilon_\sigma, \varepsilon_\tau \in \mathbf{V}\backslash(\mathbf{A} \cup \mathbf{B})$. Define the normal dissipative operator G on h by

$$G = \sum_{\sigma, \varepsilon_\sigma \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})} \left(-\frac{\mu_\sigma + \lambda_\sigma}{2} + i\kappa_\sigma\right) |\sigma\rangle\langle\sigma|$$

clearly we have

$$\pounds\left(|\sigma\rangle\langle\tau|\right) = G^*|\sigma\rangle\langle\tau| + |\sigma\rangle\langle\tau|G$$

Denoting by $(\mathrm{P}_t)_{t\geq 0}$ the strongly continuous contraction semigroup on h generated by G explicitly given by
$\mathrm{P}_t|\sigma\rangle = e^{-t(\frac{\mu_\sigma + \lambda_\sigma}{2} - i\kappa_\sigma)}|\sigma\rangle,\ for\ \varepsilon_\sigma \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B}),\ \mathrm{P}_t|\sigma\rangle = |\sigma\rangle,\ for\ \varepsilon_\sigma \in \mathbf{A}\cup\mathbf{B}$
the action of the generic QMS $(\mathcal{T}_t)_{t\geq 0}$ on the off-diagonal operator $|\sigma\rangle\langle\tau|$

should be given, in a natural way, by

$$\mathcal{T}_t(|\sigma\rangle\langle\tau|) = \mathrm{P}_t^*|\sigma\rangle\langle\tau|\mathrm{P}_t = e^{-t(\frac{\mu_\sigma+\lambda_\sigma+\mu_\tau+\lambda_\tau}{2}+i(\kappa_\sigma-\kappa_\tau))}|\sigma\rangle\langle\tau|,$$

for all σ, τ with $\varepsilon_\sigma, \varepsilon_\tau \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})$. Let $\mathcal{F}$ be the algebra generated by the operators $|\sigma\rangle\langle\tau|$ ($\varepsilon_\sigma \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})$). clearly the norm $\overline{\mathcal{F}}$ of $\mathcal{F}$ is the $\mathcal{C}^*-$algebra of all compact operators y on h such that $\langle c_j, yc_k\rangle = 0$ for all $c_j, c_k \in \mathbf{A}\cup\mathbf{B}$ and the diagonal part $\sum_{\varepsilon_\sigma\in\mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})} y_{\sigma\sigma}|\sigma\rangle\langle\sigma|$ of an operator y in the $\mathcal{C}^*-$algebra defines a function $g(\varepsilon_\sigma) = y_{\sigma\sigma}$ in $\mathcal{C}$ since $\lim_{\varepsilon_\sigma\to a_j^-} y_{\sigma\sigma} = 0$ and $\lim_{\varepsilon_\sigma\to b_k^+} y_{\sigma\sigma} = 0$.

We identify functions $f \in \mathcal{C}$ with the corresponding multiplication (diagonal) operator and denote by $\mathcal{O}$ the closed subspace of $\overline{\mathcal{F}}$ of operators with zero diagonal part (i.e $\langle\sigma, x\sigma\rangle = 0$ for all $\sigma \in V$).

Proposition 4.1 *The operators x on h can be decomposed as $f + y$ with $f \in \mathcal{C}$ and $y \in \mathcal{O}$ from a $\mathcal{C}^*-$subalgebra $\mathcal{A}$ of $\mathcal{B}(h)$.*

Definition 4.2 *For all $t \geq 0$ let $\mathcal{T}_t$ be the linear map on $\mathcal{A}$*

$$\mathcal{T}_t(a) = \mathbf{T}_t f + \mathrm{P}_t^* y \mathrm{P}_t$$

if $a = f + y$ with $f \in \mathcal{C}$ and $y \in \mathcal{O}$.

The decomposition $a = f + y$ being unique, the definition is not ambiguous. The semigroup property is easily checked. Indeed, $\mathrm{P}_t^* y \mathrm{P}_t \in \mathcal{O}$ for all $y \in \mathcal{O}$, and the semigroup property of $(\mathrm{T})_{t\geq 0}$, $(\mathrm{P}_t)_{t\geq 0}$ entail

$$\mathcal{T}_{t+s}(a) = \mathbf{T}_{t+s} f + \mathrm{P}_{t+s}^* y \mathrm{P}_{t+s} = \mathbf{T}_t \mathrm{T}_s f + \mathrm{P}_t^* (\mathrm{P}_s^* y \mathrm{P}_s) \mathrm{P}_t = \mathcal{T}_t \mathcal{T}_s(a)$$

In order to prove that the maps $\mathcal{T}_t$ are completely positive start by the following estimate.

Lemma 4.3 *For all non-negative $f \in \mathcal{C}$, $t \geq 0$ and $\varepsilon_\sigma \in \mathbf{V} \backslash (\mathbf{A} \cup \mathbf{B})$ we have*

$$(\mathbf{T}_t f)(\varepsilon_\sigma) \geq e^{-t(\mu_\sigma + \lambda_\sigma)} f(\varepsilon_\sigma)$$

Proposition 4.4 *$(\mathcal{T}_t)_{t\geq 0}$ is a semigroup of completely positive map on $\mathcal{A}$.*

Proposition 4.5 *The semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *is strongly continuous on* $\mathcal{A}$.

We Summarize the previous results by the following.

Theorem 4.6 *There exists a strongly continuous quantum Markov semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *on the* $\mathcal{C}^*-$*algebra* $\mathcal{A}$ *whose generator* $\mathcal{L}$ *defined by*

$$Dom(\mathcal{L}) = \{a \in \mathcal{A} \mid norm - \lim_{t\to 0^+} t^{-1}(\mathcal{T}_t(a) - a) exists\}$$

$$\mathcal{L}(a) = norm - \lim_{t\to 0^+} t^{-1}(\mathcal{T}_t(a) - a)$$

coincides with the form generator (2) on $a = f + y$ *with* f *either indicator of a* $\varepsilon_\sigma \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})$ *or the indicator function of an interval* $[\varepsilon_\sigma, a_j]\cap\mathbf{V}$ *(resp* $[b_k, \varepsilon_\sigma]\cap\mathbf{V}$*) with* $a_{j-1} < \varepsilon_\sigma < a_j$ *(resp* $b_k < \varepsilon_\sigma < b_{k+1}$*) and* y *is in the linear span of rank-one operator* $|\sigma\rangle\langle\tau|$ *with* $\sigma \neq \tau$ *and* $\varepsilon_\sigma, \varepsilon_\tau \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})$.

Remark 4.7 *Note that both* $\mathcal{A}$ *does not separate normal states on* $\mathcal{B}(h)$*. Indeed,defining* $\rho_\pm = |(\sigma\pm\iota)/\sqrt{2}\rangle\langle(\sigma\pm\iota)/\sqrt{2}|$*, for any* $\varepsilon_\iota \in \mathbf{A}\cup\mathbf{B}$ *and* $\varepsilon_\sigma \in \mathbf{V}\backslash(\mathbf{A}\cup\mathbf{B})$ *one has* $tr(\rho_+ a) = tr(\rho_- a)$ *for all* $a \in \mathcal{A}$.

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